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On q -Gaussians and exchangeability

Marjorie G Hahn¹, Xinxin Jiang² and Sabir Umarov¹

¹ Department of Mathematics, Tufts University, Medford, MA 02155, USA

² Department of Mathematics and Computer Science, Suffolk University, Boston, MA 02114, USA

E-mail: marjorie.hahn@tufts.edu, jiang@mcs.suffolk.edu and sabir.umarov@tufts.edu

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Abstract

The q -Gaussian distributions introduced by Tsallis are discussed from the point of view of variance mixtures of normals and exchangeability. For each $-\infty < q < 3$, there is a q -Gaussian distribution that maximizes the Tsallis entropy under suitable constraints. This paper shows that q -Gaussian random variables can be represented as variance mixtures of normals when $q > 1$. These variance mixtures of normals are the attractors in central limit theorems for sequences of exchangeable random variables, thereby providing a possible model that has been extensively studied in probability theory. The formulation provided has the additional advantage of yielding, for each q , a process which is naturally the q -analogue of the Brownian motion. Explicit mixing distributions for q -Gaussians should facilitate applications to areas such as option pricing. The model might provide insight into the study of superstatistics.

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1. Introduction

Developments in nonextensive statistical mechanics based on an entropy proposed by Constantino Tsallis (1988) gave rise to q -Gaussian distributions which are being applied in numerous research areas, see Gell-Mann and Tsallis (2004), Boon and Tsallis (2005) and Tsallis (2009). Applications are based on experimental, computational and analytical results. By definition, for $-\infty < q < 3$, the q -Gaussian density has the form $g_q(x) = C_q[1 - (1 - q)x^2]^{1/(1-q)}$ where C_q is specified in section 2. The q -Gaussian distributions discussed in this paper are those introduced in nonextensive statistical mechanics by Tsallis (1988), written in the above form first by Alemany and Zanette (1994). They are distinct from those introduced in non-commutative probability by Bożejko *et al* (1997). Theoretical under-pinnings of the q -Gaussians are founded on a novel q -algebra (Borges 2004). The purpose of this paper is to propose a stochastic model within the framework of usual algebra that is consistent with many of the phenomena researchers are attempting to model by the

q -Gaussian distributions with tails that decay as a power of x (those with $1 < q < 3$) as well as the Gaussian distribution ($q = 1$). The model not only facilitates theoretical investigations, but helps to clarify why q -Gaussian distributions arise in certain physical applications.

For each $-\infty < q < 3$, Tsallis (1988) defines a q -Gaussian distribution to be the distribution which, under certain conditions, maximizes the Tsallis q -entropy:

$$S_q = k \frac{1 - \sum_{i=1}^W p_i^q}{q - 1},$$

where $W \in \mathbb{N}$ is the total number of possible microscopic configurations, $\{p_i\}$ are the associated configuration probabilities with $\sum_{i=1}^W p_i = 1$ and k is a conventional positive constant. When $q = 1$, the definition is understood via the limit, in which case the 1-entropy recovers the Boltzmann–Gibbs entropy. The 1-Gaussian is simply the usual Gaussian distribution. The q -Gaussian distributions have tails that decay at infinity as $|x|^{-2/(q-1)}$ when $1 < q < 3$ and have bounded support when $q < 1$.

To a large extent, prevalence of the Gaussian distributions in theory and applications stems from their roles as attractors, e.g. via the classical central limit theorem for sequences of independent, identically distributed random variables with finite second moments. The dependence structure that should lead to q -Gaussian limits has remained elusive. The phrase ‘globally correlated’ is often attached to phenomena in statistical mechanics to which Tsallis entropy is applied.

There have been efforts to understand nonextensivity and the role of q -Gaussians from a probability point of view. For example, Umarov *et al* (2008) prove q -central limit theorems using q -algebra and a notion of q -independence. Since the system developed is surprisingly neat mathematically, it is interesting to understand in the sense of usual probability the intuitive meanings of q -algebra operations and q -independence. Using usual algebra and usual notions of dependence, our paper establishes that q -Gaussians, with $1 \leq q < 3$, have a role as natural attractors since they are a subset of the possible limits in a general central limit theorem for dependent random variables. This provides an explanation for the wide occurrence of at least the q -Gaussians with $1 \leq q < 3$.

Our initial investigations were stimulated by the common features of two specific mathematical models for which it is possible to analytically verify whether or not their weak limits are q -Gaussians. Vignat and Plastino (2007) suggest a model which combines randomness in the normalizer with each i.i.d. random variable, and shows that the weak limit is q -Gaussian. Their model can be viewed differently, namely as a special case of a sequence of exchangeable random variables (remark 3.5). Marsh *et al* (2006) provides practical models by applying Leibnitz triangles, in which the triangular arrays of random variables are row-wise exchangeable; however, weak limits are not q -Gaussian (Hilhorst and Schehr 2007). Both examples are discussed further in section 2 since they motivated our consideration of exchangeability as a possible probability model that might be connected to some applications being considered using Tsallis nonextensivity theory.

From a different but related direction, this paper is also stimulated by a possible connection between the proposed model and superstatistics. Beck and Cohen (2003) introduce the concept of superstatistics for generalized Boltzmann factors derived from systems that evolve in complex environments. The expression for a superstatistic can be interpreted as a variance mixture of normals (VMON) (or scale mixture of normals) after standardization. Moreover, in a specific case, the superstatistic reduces to the Tsallis statistic (Beck 2001). VMONs are the attractors in a central limit theorem for exchangeable sequences, see Jiang and Hahn (2003). Furthermore, the reason behind studying superstatistics, such as a non-homogeneous background where a mechanical system expands, strongly suggests exchangeability in modeling the underlying

system. De Finetti's theorem (Chow and Teicher 1978) characterizes an infinite sequence of exchangeable random variables as a mixture of i.i.d. sequences, i.e. an infinite exchangeable sequence is conditionally an i.i.d. sequence. Thus, if a mechanical system evolves in a complex, non-homogeneous background with the usual central limit theorem (for i.i.d. sequences) holding in each homogeneous part of the background, then the limiting process has marginals which are mixtures of normals. Further discussion of superstatistics appears at the end of section 2.

Similar arguments can be made in finance. Viewing market movements as responses to different specific information (interest rate, macro financial data, earnings, etc), then conditioning on each specific type of information, yields a market which follows a random walk. Consequently, distributions that can model asset returns could naturally be mixtures of normals. Empirical evidence (Hall *et al* 1989, Gribbin *et al* 1992, Kon 1984) has shown that discrete VMONs fit asset returns data better than stable distributions or than the Student model. Stables have been studied more extensively than discrete VMONs. Moreover, the research literature on applications of continuous VMONs seems to be quite limited.

A drawback to the applications of VMONs has always been that the mixing distributions are not generally easy to obtain from data. Expectation maximizing (EM) algorithms are usually employed. However, when the number of Gaussians in the mixture is large, or in many cases infinite, computations become very complicated. This paper shows that q -Gaussians are VMONs when $1 \leq q < 3$ (and not when $q < 1$). Furthermore, the mixing distributions are calculated explicitly, thereby facilitating application of these models. For example, the option price obtained from a VMON is the mixture of the option prices when the variance is fixed. Therefore, if one believes that q -Gaussians are good models for financial returns, then their option prices can be calculated simply by mixing the prices obtained from the classical Black–Scholes model with the known mixing distributions. Furthermore, studying mixing distributions might provide insight into the 'background'. For instance, a belief that information drives stock prices suggests viewing the mixing distributions as models for the impact of information.

The examples discussed above are each special cases of the model proposed in this paper. Our model of q -Gaussian distributions based on exchangeability, which is restricted to $1 \leq q < 3$, is specified using *usual algebra*, rather than q -algebra, and has the following features:

- (1) specification of q -Gaussian random variables as specific variance mixtures of normals;
- (2) associated central limit theorems for dependent exchangeable random variables with the q -Gaussians as natural attractors;
- (3) associated stochastic processes which in a natural sense are the q -analogs of Brownian motions.

Within this model, as the latter two features indicate, the notion of 'global correlation' is specified as exchangeability. From this point onwards we consider the appearance of q -Gaussians only as variance mixture of normals under the exchangeability concept, not discussing their appearance from other concepts. Therefore, our q -analog of the Brownian motion will be called a q -VM Brownian motion, where the VM designates variance mixture.

1.1. Organization of the paper

Section 2 makes the connection between q -Gaussians and variance mixtures of normals as well as obtaining a complete specification of the mixing distributions, which facilitates applications.

Superstatistics are discussed as an example. Section 3 provides a detailed mathematical description of exchangeability and establishes central limit theorems for exchangeable sequences and row-wise exchangeable triangular arrays. The examples motivating our consideration of exchangeability are discussed further in this section. Our concept of the q -analogue of the Brownian motion is introduced in section 4 followed by some comparisons with a stochastic process in Borland (1998) that has q -Gaussian marginal distributions. Section 5 is the conclusion.

1.2. Conventions

Throughout the paper, Z will be used specifically for a standard normal random variable that is independent of any other random variables. $\rightarrow^{\mathcal{L}}$, \rightarrow^P , $\rightarrow^{\text{a.s.}}$ stand respectively for convergence in distribution, in probability and almost surely. It is worth re-emphasizing that all further discussion relies on usual algebra rather than q -algebra.

2. q -Gaussians and variance mixtures of normals

The q -Gaussian distributions form a one-parameter family of distributions for $-\infty < q < 3$ with densities specified by $g_q(x) = C_q[1 - (1 - q)x^2]^{1/(1-q)}$, where the normalizing constant C_q is given by

$$C_q^{-1} = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi}\Gamma(\frac{1}{1-q})}{(3-q)\sqrt{1-q}\Gamma(\frac{3-q}{2(1-q)})}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^\infty (1+y^2)^{\frac{-1}{q-1}} dy = \frac{\sqrt{\pi}\Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1}\Gamma(\frac{1}{q-1})}, & 1 < q < 3. \end{cases}$$

When $q = 1$, the expression for g_q is understood by taking limits and yields the standard normal density function. The support of the q -Gaussian is $(-\infty, \infty)$ if $1 \leq q < 3$ and the compact set $[-\frac{1}{\sqrt{1-q}}, \frac{1}{\sqrt{1-q}}]$, if $-\infty < q < 1$.

A variance or (scale) mixture of normal distributions (VMON) by definition has a characteristic function of the form $\phi(t) = \int_0^\infty \exp(-t^2u/2) dH(u)$, with H a distribution function on $[0, \infty)$, called the mixing distribution. The corresponding density is

$$f(x) = \int_0^\infty (2\pi u)^{-1/2} \exp(-x^2/(2u)) dH(u).$$

Each VMON has a representation of the form VZ where $V > 0$ a.s., and V is independent of Z . The VMONs include many commonly used distributions such as the symmetric stable distributions, the Cauchy, Laplace, double exponential, logistic, hyperbolic and Student distributions and their mixtures, plus many others (Keilson and Steutel 1974, Gneiting 1997).

The connection between the q -Gaussians and the VMON can be made by applying the theory of completely monotone functions. A function h is completely monotone on $(0, \infty)$ if and only if $(-1)^n h^{(n)}(x) \geq 0$ for $x > 0$, and $n = 0, 1, 2, \dots$

Andrews and Mallows (1974) shows that a symmetric density function $f_X(x)$ is a variance mixture of normals if and only if $f_X(\sqrt{x})$ is completely monotone. It is easy to show that $g_q(\sqrt{x})$ is completely monotone for $q > 1$ and not completely monotone for $q < 1$, which leads to the following statement.

Theorem 2.1. *q-Gaussians are variance mixtures of normals when $1 < q < 3$ and not mixtures of normals when $q < 1$.*

Since not every VMON is a q -Gaussian, it remains to identify the mixing distributions that yield the q -Gaussians for $1 < q < 3$. In Beck (2001) the q -Gaussians are identified as $\frac{1}{a}Z$ where a^2 has a χ^2 distribution with q derived from the degrees of freedom. Below we provide a direct proof using the Laplace transform technique when the inverse Laplace transform of the density is known. It illustrates a method applicable to identification of unknown mixing measures from other variance mixtures of normals as well.

Theorem 2.2. *The q -Gaussian density for $1 < q < 3$ can be expressed as the following variance mixture of normal densities:*

$$g_q(x) \equiv f_{V,Z,q} = \int_0^\infty \frac{1}{\sqrt{2\pi}v} \exp\left(\frac{-x^2}{2v^2}\right) f_V(v) dv$$

where the mixing measure

$$dH(v) = f_V(v) dv = C_{V,q} \exp\left(-\frac{1}{2(q-1)v^2}\right) v^{-\frac{2}{q-1}} dv,$$

with $C_{V,q}^{-1} = \Gamma\left(\frac{3-q}{2(q-1)}\right) \cdot \frac{1}{2} \cdot [2(q-1)]^{\frac{3-q}{2(q-1)}}$.

Proof. Let L denote the Laplace transform. Since $L(\exp(-bt) \cdot t^\alpha) = \frac{\Gamma(\alpha+1)}{(\xi+b)^{\alpha+1}}$ for $\alpha > -1$, the choice of $\alpha = \frac{1}{q-1} - 1 = \frac{2-q}{q-1}$, $b = \frac{1}{q-1}$ yields

$$L\left(\exp\left(-\frac{t}{q-1}\right) \cdot t^{\frac{2-q}{q-1}}\right) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\left(\xi + \frac{1}{q-1}\right)^{\frac{1}{q-1}}} = \frac{\Gamma\left(\frac{1}{q-1}\right)(q-1)^{\frac{1}{q-1}}}{[1+(q-1)\xi]^{\frac{1}{q-1}}}.$$

Equivalently,

$$C'_q [1 - (1-q)x^2]^{1/(1-q)} = \int_0^\infty \exp(-x^2t) dH(t),$$

where $dH(t) = \exp\left(-\frac{t}{q-1}\right) \cdot t^{\frac{2-q}{q-1}} dt$ and $C'_q = \Gamma\left(\frac{1}{q-1}\right)(q-1)^{\frac{1}{1-q}}$. Routine calculations verify the claim where the following substitutions are used: $v^2 \rightarrow v$ in the second equality, $1/v \rightarrow u$ in the third equality and $\left(\frac{1}{2}\left(x^2 + \frac{1}{q-1}\right)u \rightarrow v\right)$ in the fourth equality:

$$\begin{aligned} & \int_0^\infty \frac{1}{\sqrt{2\pi}v} \exp\left(\frac{-x^2}{2v^2}\right) f_V(v) dv \\ &= C_{V,q} \int_0^\infty \frac{1}{\sqrt{2\pi}v} \exp\left(\frac{-x^2}{2v^2}\right) \exp\left(-\frac{1}{2(q-1)v^2}\right) v^{-\frac{2}{q-1}} dv \\ &= \frac{1}{2\sqrt{2\pi}} C_{V,q} \int_0^\infty \exp\left(-\frac{1}{2}\left(x^2 + \frac{1}{q-1}\right) \cdot \frac{1}{v}\right) \left(\frac{1}{v}\right)^{\frac{q}{q-1}} dv \\ &= \frac{1}{2\sqrt{2\pi}} C_{V,q} \int_0^\infty \exp\left(-\frac{1}{2}\left(x^2 + \frac{1}{q-1}\right)u\right) u^{\frac{1}{q-1}-1} du \\ &= \frac{1}{2\sqrt{2\pi}} C_{V,q} \int_0^\infty \exp(-v) \cdot v^{\frac{1}{q-1}-1} \cdot \left(\frac{1}{2}\left(x^2 + \frac{1}{q-1}\right)\right)^{\frac{-1}{q-1}} dv \\ &= \frac{1}{2\sqrt{2\pi}} C_{V,q} \Gamma\left(\frac{1}{q-1}\right) \left(\frac{1}{2}\left(x^2 + \frac{1}{q-1}\right)\right)^{\frac{-1}{q-1}} \\ &= \frac{1}{\sqrt{2\pi}} C_{V,q} 2^{\frac{2-q}{q-1}} \Gamma\left(\frac{1}{q-1}\right) (q-1)^{\frac{1}{q-1}} (1+(q-1)x^2)^{\frac{-1}{q-1}}. \end{aligned}$$

It is not hard to verify that

$$\frac{1}{\sqrt{2\pi}} C_{V,q} 2^{\frac{2-q}{q-1}} \Gamma\left(\frac{1}{q-1}\right) (q-1)^{\frac{1}{q-1}} = C_q,$$

which completes the proof. □

In theorem 2.2, V is 1 over the square root of a χ^2 distribution with the number of degrees of freedom being $\frac{2}{q-1} - 1$. Also note that when $q = 2$, the q -Gaussian is Cauchy and

$$f_V(v) = \sqrt{\frac{2}{\pi}} \frac{1}{v^2} \exp\left(\frac{-1}{2v^2}\right).$$

When $q < 4/3$, $EX^2 < \infty$. This situation is important in finance or risk management where the variance is often used to quantify the risk. In that situation, the q -Gaussians are superior to non-Gaussian stable distributions which fail to have finite second moments. When $q < 3/2$, $EX < \infty$. This situation is important in finance or insurance when measuring mean returns or mean expenses.

The mixing distributions are a type of generalized inverse Gaussian distribution, and the q -Gaussians are a type of generalized hyperbolic distribution. It is worth noting that both generalized inverse Gaussian and generalized hyperbolic distributions are infinitely divisible. Since infinitely divisible laws are the only attractors for triangular arrays of independent random variables, this fact might help to explain why q -Gaussians should be often observed.

2.1. Example: superstatistics

As a test particle moves from cell to cell, its velocity v satisfies a Langevin equation

$$\dot{v} = -\gamma v + \sigma \dot{B}_t, \tag{1}$$

where B_t is a standard Brownian motion. Instead of γ and σ being deterministic, as in classical statistical physics, Beck and Cohen (2003) let $\beta = \gamma/\sigma^2$ be a random variable with density $f(\beta)$. In this setting, the widely used classical Boltzman factor $e^{-\beta E}$ takes a generalized form called a superstatistic:

$$B(E) = \int_0^\infty k(\beta) e^{-\beta E} d\beta, \tag{2}$$

where E represents the energy of a microstate associated with each cell. Note that the superstatistic in (2) is exactly a variance mixture of normals after standardization. The special case where $k(\beta)$ (the density of $1/V^2$ in our theorem 2.2) is the density of a χ^2 random variable yields the Tsallis statistic (Beck 2001).

3. Central limit theorems for exchangeable random variables

A brief introduction to exchangeability is required (for details, see Chow and Teicher 1978).

A sequence of random variables $\{X_n, n \geq 1\}$ defined on some probability space is said to be *exchangeable* if for each n ,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = P(X_{\pi(1)} \leq x_1, \dots, X_{\pi(n)} \leq x_n)$$

for any permutation π of $\{1, 2, \dots, n\}$ and any $x_i \in \mathbf{R}, i = 1, \dots, n$.

By de Finetti's theorem, an infinite sequence of exchangeable random variables in some appropriate space is conditionally i.i.d., given the σ -field \mathcal{G} of permutable events. Furthermore, there exists a regular conditional distribution P^ω for X_i , given \mathcal{G} such that for each $\omega \in \Omega$

the coordinate random variables $\{\xi_n \equiv \xi_n^\omega, n \geq 1\}$, called *mixands*, are i.i.d. Hence, for each natural number n , any Borel function $f : \mathbf{R}^n \rightarrow \mathbf{R}$, and any Borel subset B of \mathbf{R} ,

$$\begin{aligned} P(f(X_1, \dots, X_n) \in B) &= \int_{\Omega} P(f(X_1, \dots, X_n) \in B | \mathcal{G}) dP \\ &= \int_{\Omega} P^\omega(f(\xi_1, \dots, \xi_n) \in B) dP. \end{aligned}$$

The mixands are allowed to be on a different probability space than the X_i 's.

For exchangeable sequences, the dependence never dies in contrast to weakly dependent sequences. Hence if the covariance exists, it does not change along the sequence. Moreover, for an infinite exchangeable sequence the covariance is always non-negative.

3.1. Example

The model proposed in Marsh *et al* (2006) considers N identical and distinguishable, but not necessarily independent binary subsystems. Let $r_{N,n}$ be the probability that there are n subsystems in state 1, which is given by the Leibnitz rule:

$$r_{N,n} + r_{N,n+1} = r_{N-1,n}.$$

Since construction of the model only considers the number of subsystems in state 1, the order of 1's and 0's does not matter, a typical property of exchangeable sequences. Hilhorst and Schehr (2007) showed that the weak limit is not a q -Gaussian. However, using the same Leibnitz rule, Rodriguez *et al* (2008) and Hanel *et al* (2009) constructed exchangeable models that do have a q -Gaussian limit.

Turning to central limit theorems, the following simplified and adapted version of theorem 2.1 in Jiang and Hahn (2003) suffices for the needs of this paper. The proof is provided for completeness.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be an infinite sequence of exchangeable random variables where X_1 has mean 0 if the mean exists or is symmetric otherwise. Assume that $0 < E \xi_1^2 < \infty$ a.s. Then either*

$$(i) \quad \frac{\sum_{i=1}^n X_i}{\sqrt{n}} \rightarrow^{\mathcal{L}} V_1 \cdot Z$$

or

$$(ii) \quad \frac{\sum_{i=1}^n X_i}{n} \rightarrow^{a.s.} V_2,$$

where $V_1 = \sqrt{\text{Var}(\xi_1)} = \sqrt{\text{Var}(X_1 | \mathcal{G})}$ and $V_2 = E \xi_1 = E(X_1 | \mathcal{G})$.

Proof. By de Finetti's theorem, for any real x ,

$$P\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq x\right) = \int_{\Omega} P^\omega\left(\frac{\sum_{i=1}^n \xi_i - nE\xi_1}{\sqrt{n} \cdot \sigma(\xi_1)} \sigma(\xi_1) + \sqrt{n}E\xi_1 \leq x\right) dP,$$

where $\sigma(\xi_1)$ denotes the standard deviation of ξ_1 . If $E\xi_1 = 0$ a.s., then the classical central limit theorem holds for the mixands and case (i) of the theorem holds. If $E\xi_1$ is not almost surely 0, replace the \sqrt{n} in the previous equation by n . Then the first summand inside the regular conditional probability on the right side integral converges to zero almost surely and case (ii) of the theorem holds. \square

Remark 3.2. The condition put on the mixands, $0 < E\xi_1^2 < \infty$ a.s., is generally weaker than assuming that X_i 's have finite second moments.

Remark 3.3. Case (i) of theorem 3.1 shows that q -Gaussians when $1 < q < 3$ are among the possible limits in a central limit theorem for exchangeable sequences. However, they are not the only ones. It seems more natural to consider q -Gaussians as limit distributions in case (i) since theoretically the range of distributions in case (ii) is vast. For example, let ϵ_i 's be i.i.d. with mean 0 and standard deviation 1, and Y be any random variable that is independent of all ϵ_i 's. Then $\{Y + \epsilon_i, i \geq 1\}$ is a sequence of exchangeable random variables and $\sum(Y + \epsilon_i)/n \rightarrow^{a.s.} Y$.

Remark 3.4. In case (i), $EX_1X_2 = 0$ if it exists, which means that q -Gaussians can be attractors of uncorrelated, but not necessarily independent, exchangeable sequences of random variables. In the second case, $EX_1X_2 \neq 0$ if it exists and in this case the central limit theorem is really the strong law of large numbers for exchangeable random variables. This case explains why the examples in Rodriguez *et al* (2008) and Hanel *et al* (2009) achieve q -Gaussian limits with normalizers at the rate of n .

Remark 3.5. Theorem 4.1 of Vignat and Plastino (2007) says that if Y_i are i.i.d. in the domain of normal attraction of Z and a^2 has a χ^2 -distribution (correcting a typo), then $\frac{1}{a\sqrt{n}} \sum_1^n Y_i$ converges in distribution to a q -Gaussian distribution. Their theorem in one dimension is a special case of theorem 3.1 in this paper with $X_i = V_1 \cdot Y_i$ where $V_1 = \frac{1}{a}$. However, in general, a sequence of exchangeable random variables does not necessarily have the form $V \cdot Y_i$.

The next theorem is a triangular array version of theorem 3.1, which seems to be new.

Theorem 3.6. Let $\{X_{n,i}, 1 \leq i \leq n, n = 1, 2, \dots\}$ be a triangular array of row-wise exchangeable random variables that can be embedded into infinite exchangeable sequences. Assume that $X_{n,i}$'s are centered or symmetric with $0 < E \xi_{n,1}^2 < \infty$ in probability for each n , and $Var(\xi_{n,1}) \rightarrow V_1^2$ in probability when $n \rightarrow \infty$, with $V_1 > 0$ a.s. Then

$$\frac{\sum_{i=1}^n X_{n,i}}{\sqrt{n}} \rightarrow^{\mathcal{L}} V_1 \cdot Z + V_2$$

when $\sqrt{n}E\xi_{n,1} \rightarrow V_2$, and

$$\frac{\sum_{i=1}^n X_{n,i}}{n^\alpha} \rightarrow^P V_3$$

with $\alpha > 1/2$, when $n^{1-\alpha}E\xi_{n,i} \rightarrow^P V_3$ as $n \rightarrow \infty$.

Proof. By de Finetti's theorem, for any real x ,

$$P\left(\frac{\sum_{i=1}^n X_{n,i}}{\sqrt{n}} \leq x\right) = \int_{\Omega} P^\omega\left(\frac{\sum_{i=1}^n \xi_{n,i} - nE\xi_{n,1}}{\sqrt{n} \cdot \sigma(\xi_{n,1})} \sigma(\xi_{n,1}) + \sqrt{n}E\xi_{n,1} \leq x\right) dP.$$

Using the definition of weak convergence in probability (Hahn and Zhang 1998), and the conditions that $\sqrt{n}E\xi_{n,1} \rightarrow^P V_2$, $\sigma(\xi_{n,1}) \rightarrow^P V_1$ with $V_1 > 0$ a.s. and that the mixands have finite second moments almost surely, yields, when $n \rightarrow \infty$,

$$P\left(\frac{\sum_{i=1}^n X_{n,i}}{\sqrt{n}} \leq x\right) \rightarrow \int_{\Omega} P^\omega(V_1Z + V_2 \leq x) dP = P(V_1Z + V_2 \leq x).$$

When $n^{1-\alpha}E\xi_{n,1} \rightarrow^P V_3$ with $\alpha > 1/2$ and $V_2 \neq 0$ a.s., again, by de Finetti's theorem,

$$P\left(\frac{\sum_{i=1}^n X_{n,i}}{n^\alpha} \leq x\right) = \int_{\Omega} P^\omega\left(\frac{\sum_{i=1}^n \xi_{n,i} - nE\xi_{n,1}}{n^\alpha \cdot \sigma(\xi_{n,1})} \sigma(\xi_{n,1}) + n^{1-\alpha}E\xi_{n,1} \leq x\right) dP.$$

Since

$$\frac{\sum_{i=1}^n \xi_{n,i} - nE\xi_{n,1}}{n^\alpha \sigma(\xi_{n,1})} \rightarrow^P 0$$

when $\alpha > 1/2$ and $\sigma(\xi_{n,1}) \rightarrow^P V_2$,

$$\frac{\sum_{i=1}^n \xi_{n,i} - nE\xi_{n,1}}{n^\alpha \cdot \sigma(\xi_{n,1})} \sigma(\xi_{n,1}) \rightarrow^P 0,$$

and the second part of the theorem is proved. □

Remark 3.7. When $X_{n,i}$'s are identically distributed, then $V_2 = 0$ and $\alpha = 1$, which is consistent with theorem 3.1.

4. q -analog of the Brownian motion

The literature on q -Gaussians thus far has failed to provide the construction of a clear q -analog of the Brownian motion. Our representation of q -Gaussians for $q > 1$ in terms of variance mixtures of normals, using usual probabilistic and algebraic notions, leads to the definition of a process that might naturally be called a q -Brownian motion. However, that name is already in use in other areas. Thus, the process to be constructed will be called a q -VM Brownian motion, where VM reflects the fact that it is a variance mixture of Brownian motions.

We first require some definitions.

Definition 4.1. A stochastic process is called exchangeable if it is continuous in probability with $X_0 = 0$ and such that the increments over disjoint intervals of equal length form an exchangeable sequence.

Definition 4.2. X has conditionally independent, stationary increments, given some σ -field, if both properties of the increments are conditionally valid for any finite collection of disjoint intervals of the same length.

These two definitions are connected by the following theorem of Bühlmann which characterizes exchangeable processes on R^+ (see e.g. theorem 9.21 of Kallenberg (2002)).

Theorem 4.3 (Bühlmann). Let the process $(X_t)_{t \geq 0}$ be R^d -valued and continuous in probability with $X_0 = 0$. Then X is exchangeable if and only if it has conditionally independent, stationary increments given some σ -field.

We can now define our q -VM Brownian motion.

Definition 4.4. A q -VM Brownian motion, B_t^q , with $1 \leq q < 3$ is a stochastic process having the following properties:

- (1) conditionally independent, stationary increments;
- (2) all increments are q -Gaussian;
- (3) a.s. continuous sample paths.

Theorem 4.5 (Existence). For each $1 \leq q < 3$, a q -VM Brownian motion exists.

Proof. Let B_t be a standard Brownian motion. When $q = 1$, $B_t^1 = B_t$ is a process satisfying all three conditions with the conditional σ -field being the trivial σ -field. For $1 < q < 3$, let V_q be a random variable with the density f_q given in theorem 2.2. Then $B_t^q = V_q \cdot B_t$ has conditionally stationary and independent increments given the value of V_q . All increments are q -Gaussian. Furthermore, a variance mixture of processes with a.s. continuous sample paths has a.s. continuous sample paths. □

4.1. Example: comparison with the Borland process

Borland (1998) provides another process with q -Gaussian marginals. Let Y_t be the log returns of stock prices that follow the stochastic differential equation $dY_t = \mu dt + \sigma d\Omega_t$, where Ω_t evolves according to $d\Omega_t = P(\Omega_t)^{(1-q)/2} dB_t$. The evolution of the probability distribution P of Y_t is nonlinear according to the nonlinear Fokker–Planck equation ($\mu = 0, \sigma = 1$)

$$\frac{\partial}{\partial t} P(x, t|y) = \frac{1}{2} \frac{\partial^2}{\partial x^2} [P^{2-q}(x, t|y)].$$

Solutions for $P(x, t|y)$ are given by q -Gaussians for each fixed t , as established in Borland (1998) or using the q -Fourier transform as in Umarov and Queiros (2009).

Our q -VM Brownian motion has conditionally, stationary and independent increments, each of which is q -Gaussian. This is expressed by the fact that its transition density is expressed in the form

$$P(t, x|y) = \int_0^\infty p(t, x|y, v) f_V(v) dv,$$

where $p(t, x|y, v)$ for each fixed v satisfies the linear Fokker–Planck equation

$$\frac{d}{dt} p(t, x|y, v) = \frac{1}{2} v^2 \frac{\partial^2 p(t, x|y, v)}{\partial x^2},$$

and the occurrence of the transition probability for a particular v is weighted according to the density $f_V(v)$ specified in section 2.

The Borland process clearly differs from our q -VM Brownian motion. Even though the Borland process has stationary increments (Umarov and Queiros 2009), its increments are not conditionally independent. It does have continuous paths and q -Gaussian marginals.

5. Conclusion

This paper provides a probabilistic model for q -Gaussian distributions with $1 \leq q < 3$ based on exchangeability as one of the possible notions of ‘global correlation’. The model is consistent with many of the phenomena for which q -Gaussian distributions are being used based on empirical evidence. In particular, q -Gaussian distributions are variance mixtures of normals when $1 \leq q < 3$ and not when $q < 1$. Explicit mixing distributions are provided, which should extend further application of q -Gaussian distributions. An explanation for the wide occurrence of these q -Gaussians is their role as attractors via central limit theorems for exchangeable sequences and triangular arrays. A natural q -analog of the Brownian motion is defined, the q -VM Brownian motion, which can be viewed as an alternative driving process to the Borland (1998) process. The increments of the two processes have different characteristics and thus model different phenomena. The paper also makes a connection with superstatistics which are variance mixtures of normals after normalization. The Langevin equations that yield superstatistics can be viewed as stochastic differential equations driven by exchangeable processes.

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